[Turkish Journal of Mathematics](https://journals.tubitak.gov.tr/math)

[Volume 45](https://journals.tubitak.gov.tr/math/vol45) | [Number 4](https://journals.tubitak.gov.tr/math/vol45/iss4) Article 7

1-1-2021

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Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2021) 45: 1580 – 1591
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doi:10.3906/mat-2009-30

Logarithmic dimension and bases in Whitney spaces

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Received: 07.09.2020 • **Accepted/Published Online:** 02.05.2021 • **Final Version:** 27.07.2021

Abstract: We give a formula for the logarithmic dimension of the generalized Cantor-type set K. In the case when the logarithmic dimension of K is smaller than 1, we construct a Faber basis in the space of Whitney functions $\mathcal{E}(K)$.

Key words: Topological bases, Whitney spaces, Hausdorff dimension, logarithmic capacity

1. Introduction

This paper is the extension of [2] and [12]. In [2], the logarithmic dimension λ_0 was suggested as the Hausdorff dimension corresponding to the function $\psi(r) = \frac{1}{\log \frac{1}{r}}$ that defines the logarithmic measure. Some applications of the logarithmic dimension to the isomorphic classification of Whitney spaces were presented. In [12], the first author constructed bases in the spaces $\mathcal{E}(K_2^{(\alpha_n)})$, where the set $K_2^{(\alpha_n)}$ is obtained by the Cantor procedure with replacing each interval by two *adjacent* subintervals of equal length. Here, as in [2], we consider more general Cantor-type sets $K_{(N_n)}^{(\alpha_n)}$, see the definition below. In Section 2, we generalize Proposition 1 from [2], where the logarithmic dimension was calculated for regular $K_{(N_n)}^{(\alpha_n)}$. In Sections 3 we discuss applications of the logarithmic dimension to potential theory and to analysis of linear topological properties of Whitney spaces. Section 4 is devoted to construction of an interpolating Faber basis in $\mathcal{E}(K_{(N_n)}^{(\alpha_n)})$ provided $\lambda_0(K_{(N_n)}^{(\alpha_n)}) < 1$.

2. Logarithmic dimension for the generalized Cantor-type sets

Recall that a function $\varphi : (0, b] \to (0, \infty)$, where $b = b_\varphi > 0$, is said to be a dimension function if it is nondecreasing, continuous and $\varphi(\delta) \to 0$ as $\delta \to 0$. Given $A \subset \mathbb{R}$, $\varepsilon > 0$, let $\mu_\varepsilon(A, \varphi) = \inf\{\sum \varphi(\delta_i) : A \subset \cup G_i \text{ with } \operatorname{diam}(G_i) = \delta_i \le \varepsilon\}$. Here, the infimum can be taken over open coverings or closed coverings without changing the result. The value $\mu_\varepsilon(A, \varphi)$ increases as $\varepsilon \searrow 0$ and $\mu(A, \varphi) = \lim_{\varepsilon \to 0} \mu_\varepsilon(A, \varphi)$ is called the Hausdorff $\varphi-$ measure of A.

Logarithmic dimension is a special case of the Hausdorff dimension. Take the function $\psi(r) = \frac{1}{\log \frac{1}{r}}$ corresponding to the logarithmic measure. Then, for any $A \subset \mathbb{R}$ there exists a critical value $\lambda_0 = \lambda_0(A) \in [0, \infty]$, which we call the *logarithmic dimension of* A, such that for $\lambda < \lambda_0$ the Hausdorff ψ^λ-measure of A is infinite,

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2010 *AMS Mathematics Subject Classification:* 46A35, 46E10, 41A10

and for $\lambda > \lambda_0$ it is zero. As usual, the ψ^{λ_0}-measure of A can take any value from $[0, +\infty]$.

We follow [2] to define generalized Cantor-type sets. Let $(N_n)_{n=1}^{\infty}$ be a sequence of integers with $N_n \geq 2$ for all n. Let $\ell_0 := 1$ and ℓ_1 be such that $N_1 \ell_1 < \ell_0$. We replace $E_0 = I_{0,1} = [0,1]$ by N_1 closed intervals $I_{n,1}$ of length ℓ_1 with $N_1 - 1$ equal gaps of length h_0. We enumerate intervals in ascending order, so $I_{1,1} = [0, \ell_1], I_{N_1,1} = [1 - \ell_1, 1]$. Continuing in this way, we get E_n for $n \geq 1$ as a union of $N_1 N_2 \dots N_n$ disjoint closed intervals $I_{k,n}$ of length ℓ_n, and E_{n+1} is obtained by replacing each interval $I_{k,n}$ by N_{n+1} disjoint subintervals $I_{j,n+1}$ of length ℓ_{n+1} with $N_{n+1} - 1$ equal gaps of length h_n. The intervals $I_{k,n}$ that make up the set E_n are called *basic intervals*. The set is well-defined if for all n we have $N_n \ell_n < \ell_{n-1}$. Then $h_n = \frac{\ell_n - N_{n+1}\ell_{n+1}}{N_{n+1}-1}$ is a gap between To simplify the calculation of the norms, assume that for each n

$$h_n \geq \ell_{n+1}. \tag{2.1}$$

Thus, we get a sequence $(\ell_n)_{n=0}^{\infty}$ of positive decreasing numbers. Let $\alpha_1 = 1$, and for $n \geq 2$ let α_n satisfy $\ell_n = \ell_{n-1}^{\alpha_n}$, so $\alpha_n > 1$. Thus, $\ell_n = \ell_1^{\alpha_1 \cdots \alpha_n}$. Let $K_{(N_n)}^{(\alpha_n)} := \bigcap_{n=0}^{\infty} E_n$. We will denote by K_N^{α} the case when $N_n = N$ and $\alpha_n = \alpha$, for all indices.

Lemma 2.1 *For each $K_{(N_n)}^{(\alpha_n)}$ we have $\alpha_1 \cdots \alpha_n \to \infty$ as $n \to \infty$.*

Proof The sequence $(\alpha_1 \cdots \alpha_n)_{n=1}^{\infty}$ increases. If it is bounded, then $\alpha_n \to 1$ as $n \to \infty$. But $N_n \ell_n < \ell_{n-1}$ implies $N_n \ell_1^{\alpha_1 \cdots \alpha_{n-1}(\alpha_n - 1)} < 1$, a contradiction. □

We say that the Cantor-type set $K_{(N_n)}^{(\alpha_n)}$ is regular if there exists $\lim_n \frac{\log N_n}{\log \alpha_n}$. The logarithmic dimension of a regular Cantor-type set was given in [2] as follows:

Proposition 2.2 *Suppose that for $K_{(N_n)}^{(\alpha_n)}$ the limit $\lambda_0 = \lim_n \frac{\log N_n}{\log \alpha_n}$, exists in the set of extended real numbers. Then λ_0 is the logarithmic dimension of K. In particular, $\lambda_0(K_N^{\alpha}) = \frac{\log N}{\log \alpha}$.*

We now extend this result to the general case. The proof is adapted from [2].

Theorem 2.3 *For the generalized Cantor-type set $K_{(N_n)}^{(\alpha_n)}$, we have*

$$\lambda_0(K_{(N_n)}^{(\alpha_n)}) = \liminf_n \frac{\log(N_1 N_2 \dots N_n)}{\log(\alpha_1 \alpha_2 \dots \alpha_n)}.$$

Proof As above, $\psi = \frac{1}{\log \frac{1}{r}}$ for $0 < r < 1$ and, for a given $\lambda > 0$, let $\mu(K, \psi^{\lambda})$ be the Hausdorff $\psi^{\lambda}-$measure of K. For simplicity of notation and calculations, we write K instead of a fixed $K_{(N_n)}^{(\alpha_n)}$ and set $\ell_1 = 1/e$ in order to have $\psi(\ell_1) = 1$. Then $\psi^{\lambda}(\ell_n) = (\alpha_1 \alpha_2 \dots \alpha_n)^{-\lambda}$. Define $\lambda_n = \frac{\log(N_1 N_2 \dots N_n)}{\log(\alpha_1 \alpha_2 \dots \alpha_n)}$ for $n \geq 2$. Then

$$(\alpha_1 \alpha_2 \dots \alpha_n)^{\lambda_n} = N_1 N_2 \dots N_n. \tag{2.2}$$

Let $\lambda_0 = \liminf_n \lambda_n$. We claim that $\lambda_0 = \lambda_0(K)$.

There are two cases to consider: finite and infinite λ_0. Suppose first that $0 \le \lambda_0 < \infty$ and $\lambda > \lambda_0$. Let $\lambda = \lambda_0 + 2\sigma$. We need to show that $\mu(K, \psi^\lambda) = 0$.

By definition, there exists $n_k \to \infty$ such that $\lambda_0 = \lim_k \lambda_{n_k}$ so $\lambda > \lambda_{n_k} + \sigma$ for large enough k. Since E_n is a covering of K by $N_1 \dots N_n$ intervals of length ℓ_n, by (2.2), we have

$$\mu(K, \psi^\lambda) \le \liminf_n (N_1 \dots N_n)\psi^\lambda(\ell_n) = \liminf_n \frac{N_1 \dots N_n}{(\alpha_1 \dots \alpha_n)^\lambda} =$$

$$= \liminf_n (\alpha_1 \dots \alpha_n)^{\lambda_n - \lambda} \le \liminf_k (\alpha_1 \dots \alpha_{n_k})^{\lambda_{n_k} - \lambda} \le \liminf_k (\alpha_1 \dots \alpha_{n_k})^{-\sigma}.$$

By Lemma 2.1, the above limit is zero.

We now turn to the case $0 < \lambda_0 < \infty$ and $\lambda < \lambda_0$. We aim to show $\mu(K, \psi^\lambda) = \infty$. Let $\lambda_0 - \lambda = 2\sigma$. There are only finitely many n with $\lambda_n \le \lambda_0 - \sigma$. Let $\tilde{n}$ be such that $\lambda_n > \lambda_0 - \sigma$ for $n \ge \tilde{n}$. Then $\lambda_n > \lambda + \sigma$.

We fix $\epsilon > 0$ and consider $\mu_\epsilon(K, \psi^\lambda)$. Here we use coverings of K by open intervals. Let us fix a finite covering $\bigcup_{i=1}^M G_i$ of K by open intervals with lengths $\delta_i < \epsilon$, such that

$$\sum_{i=1}^{M} \psi^\lambda(\delta_i) \le \mu_\epsilon(K, \psi^\lambda) + 1. \tag{2.3}$$

For each δ_i fix $n = n(i) \in \mathbb{N}$ with $\ell_n \le \delta_i < \ell_{n-1}$. Let $n_0 = \min_{i \le M} n(i)$ and $n_1 = \max_{i \le M} n(i)$. We can assume, by decreasing ϵ if necessary, that $n_0 \ge \tilde{n} + 1$.

For $1 \le i \le M$, let k_i be the number of intervals from E_{n_1} that have non-empty intersection with G_i. We follow [10] and [2], where the main idea was to estimate k_i from above in terms of $\psi^\lambda(\delta_i)$.

For each i we have $\psi^\lambda(\delta_i) \ge \psi^\lambda(\ell_n) = (\alpha_1 \alpha_2 \cdots \alpha_n)^{-\lambda}$. Since $\lambda_n > \lambda$ for $n \ge n_0$, (2.2) implies

$$(\alpha_1 \cdots \alpha_n)^\lambda < (\alpha_1 \cdots \alpha_{n_0 - 1})^\lambda \cdot (\alpha_{n_0} \cdots \alpha_n)^{\lambda_n} = (\alpha_1 \cdots \alpha_{n_0-1})^{\lambda - \lambda_n} \cdot N_1 N_2 \cdots N_n. \tag{2.4}$$

Therefore,

$$1 \le (\alpha_1 \cdots \alpha_{n_0 - 1})^{\lambda - \lambda_n} \cdot N_1 N_2 \cdots N_n \cdot \psi^\lambda(\delta_i). \tag{2.5}$$

In what follows we will use (2.4) with another index, $n-1$ instead of n. The left hand side of (2.4) exceeds 1. Hence,

$$1 \le (\alpha_1 \cdots \alpha_{n_0-1})^{\lambda - \lambda_{n-1}} \cdot N_1 N_2 \cdots N_{n-1}. \tag{2.6}$$

We decompose the sum $\sum \psi^\lambda(\delta_i)$ into two parts. Let $\sum'$ be the sum over all i such that $\ell_n \le \delta_i < \frac{\ell_{n-1}}{N_n}$, and $\sum''$ be the sum over the remaining i's. Since $\frac{\ell_{n-1}}{N_n} < \ell_n + h_{n-1}$, for any i in the sum $\sum'$, the interval G_i can intersect at most two basic intervals of E_n. By construction, it can intersect at most $2N_{n+1}$ basic intervals of $E_{n+1}, \dots, 2N_{n+1} \cdots N_{n_1}$ basic intervals of E_{n_1} .

Then by (2.5) we obtain for each i corresponding to $\sum'$

$$k_i \le 2N_{n+1} \cdots N_{n_1} \le 2N_1 \cdots N_{n_1} \cdot (\alpha_1 \cdots \alpha_{n_0-1})^{\lambda - \lambda_n} \cdot \psi^\lambda(\delta_i). \tag{2.7}$$

For i corresponding to $\sum''$ we fix $j \in \{1, 2, \dots, N_n - 1\}$ such that $\frac{j}{N_n}\ell_{n-1} \le \delta_i < \frac{j+1}{N_n}\ell_{n-1}$. It is easy to check that the interval G_i can intersect at most $j+2$ basic intervals of E_n and hence $(j+2)N_{n+1} \cdots N_{n_1}$ basic intervals of E_{n_1}.

Here,

$$\psi^\lambda(\delta_i) \geq \psi^\lambda\left(\frac{j}{N_n}\ell_{n-1}\right) \geq \left(\alpha_1 \ldots \alpha_{n-1} + \log\frac{N_n}{j}\right)^{-\lambda}.$$

If $\log\frac{N_n}{j} \geq \alpha_1 \cdots \alpha_{n-1}$, then $\psi^\lambda(\delta_i) \geq (2\log\frac{N_n}{j})^{-\lambda}$. Recall that $1 < \frac{N_n}{j} \leq N_n$. Take a constant A_λ such that $\log^\lambda t \leq A_\lambda t$ for $t \geq 1$. Then $1 \leq 2^\lambda A_\lambda \frac{N_n}{j}\psi^\lambda(\delta_i)$ and

$$k_i \leq (j+2)N_{n+1} \cdots N_{n_1} \leq 2^\lambda A_\lambda \frac{j+2}{j} N_n N_{n+1} \cdots N_{n_1} \psi^\lambda(\delta_i).$$

Here, $\frac{j+2}{j} \leq 3$. Let $C'_\lambda = 3 \cdot 2^\lambda A_\lambda$. By (2.6),

$$k_i \leq C'_\lambda (\alpha_1 \cdots \alpha_{n_0-1})^{\lambda-\lambda_{n-1}} \cdot N_1 \cdots N_{n_1}\psi^\lambda(\delta_i). \tag{2.8}$$

Suppose now that $\log\frac{N_n}{j} < \alpha_1 \cdots \alpha_{n-1}$. Then $\psi^\lambda(\delta_i) \geq (2\,\alpha_1 \ldots \alpha_{n-1})^{-\lambda}$. Since $j+2 \leq N_n + 1 < 2\,N_n$, we have

$$k_i \leq 2N_n \cdots N_{n_1} \leq 2^{\lambda+1}(\alpha_1 \ldots \alpha_{n-1})^\lambda N_n \cdots N_{n_1}\psi^\lambda(\delta_i).$$

By (2.4),

$$k_i \leq 2N_n \cdots N_{n_1} \leq 2^{\lambda+1}(\alpha_1 \cdots \alpha_{n_0-1})^{\lambda-\lambda_n} N_1 \cdots N_{n_1}\psi^\lambda(\delta_i).$$

Combining this with (2.7) and (2.8), we see that for each i, the inequality

$$k_i \leq C_\lambda (\alpha_1 \cdots \alpha_{n_0-1})^{-\sigma} N_1 \cdots N_{n_1}\psi^\lambda(\delta_i)$$

is valid with $C_\lambda = \max\{C'_\lambda, 2^{\lambda+1}\}$. Here we use the conditions $\lambda_k - \lambda > \sigma$ for $k \in \{n-1, n\}$.

The covering $\bigcup_{i=1}^M G_i$ intersects all basic intervals of E_{n_1}, so $\sum_{i=1}^M k_i \geq N_1 \cdots N_{n_1}$. This gives

$$C_\lambda^{-1}(\alpha_1 \cdots \alpha_{n_0-1})^\sigma \leq \sum_{i=1}^M \psi^\lambda(\delta_i). \tag{2.9}$$

By Lemma 2.1, the left hand side here is as big as we want for small enough ϵ. By (2.3), $\mu_\epsilon(K, \psi^\lambda) \to \infty$ as $\epsilon \to 0$, which is our claim.

It remains to consider the case of infinite λ_0. Fix any λ. We repeat the previous arguments with minor modifications. Here, $\tilde{n}$ is given by the condition $\lambda_n \geq 2\lambda$ for $n \geq \tilde{n}$. In the same manner we get (2.9) with λ instead of σ and $\mu(K, \psi^\lambda) = \infty$. □

Remarks. 1. A set K is called *dimensional* if there is at least one dimension function φ such that $0 < \mu(K, \varphi) < \infty$. Best in [4] presented an example of a dimensionless Cantor set. The theorem above does not mean that each sets $K = K^{(\alpha_n)}_{(N_n)}$ is dimensional, because the value $\mu(K, \psi^{\lambda_0})$ may be 0 or ∞. Nevertheless, we think that for every K of the given type, there is a function φ (possibly more complex in structure than ψ^λ) with a proper value of $\mu(K, \varphi)$. See for instance [1] for the construction of such function for a more complicated Cantor-type set that is not geometrically symmetric.

2. In the proof we did not use the condition (2.1).

3. Relation to potential theory and the extension property

The value $\lambda_0 = 1$ is critical in potential theory: by Theorem III.19 and Theorem III.20 in [16], we have the following simple observation.

Proposition 3.1 *Assume* $\lambda_0 = \lambda_0(K_2^{(\alpha_n)}) \neq 1$. *Then* $K_2^{(\alpha_n)}$ *is polar if and only if* $\lambda_0 < 1$.

In the case of $\lambda_0(K) = 1$, the finiteness of the logarithmic measure is sufficient for polarity.

Proposition 3.2 *([9]) If* $\mu(K, \psi) < \infty$ *then* $Cap(K) = 0$.

By Carleson [5] (see also [6]), we have

Proposition 3.3 *The set* $K_2^{(\alpha_n)}$ *is polar if and only if* $\sum_{n=1}^{\infty} \frac{A_n}{2^n} = \infty$, *where* $A_n = \alpha_1 \alpha_2 \dots \alpha_n$.

It is easy to give examples of both polar and non-polar Cantor sets of logarithmic dimension 1. Let $K_1 := K_2^{(\alpha_n)}$ with $A_n = 2^n/n^2$ for large n and $K_2 := K_2^2$. Then $Cap(K_1) > 0,\ Cap(K_2) = 0,\ \lambda_0(K_1) = \lambda_0(K_2) = 1$.

Also, the example $K_2^{(\alpha_n)}$ with $\alpha_2 = 2$ and $\alpha_n = 2\frac{n-1}{n}, n \geq 3$ (here, $A_n = 2^n/n$) shows that the inverse implication in Propositions 3.2 is not valid.

Let $K \subset \mathbb{R}$ be a perfect compact set and I be a closed interval containing K. By $\mathcal{F}(K, I) = \{F \in C^\infty(I) : F^{(p)}|_K = 0,\ \forall p\}$ we denote the ideal of flat on K functions. The Whitney space $\mathcal{E}(K)$ of extendable functions consists of traces on K of C^∞-functions defined on I, so it is a factor space of $C^\infty(I)$ and the restriction operator $R : C^\infty(I) \longrightarrow \mathcal{E}(K)$ is surjective. This means that the sequence $0 \longrightarrow \mathcal{F}(K, I) \xrightarrow{J} C^\infty(I) \xrightarrow{R} \mathcal{E}(K) \longrightarrow 0$ is exact. If it splits, then the right inverse to R is the linear continuous extension operator $W : \mathcal{E}(K) \longrightarrow C^\infty(I)$. In this case we say that K has the extension property.

By the celebrated Whitney theorem ([18]), the quotient topology of $\mathcal{E}(K)$ can be given by the norms

$$\|f\|_q = |f|_q + \sup\{|(R_y^q f)^{(i)}(x)| \cdot |x - y|^{i-q} : x, y \in K, x \neq y, i = 0, 1, \dots, q\},$$

where $q = 0, 1, \dots,\ |f|_q = \sup\{|f^{(i)}(x)| : x \in K, i \leq q\}$ and $R_y^q f(\cdot) = f(\cdot) - \sum_{k=0}^{q} \frac{f^{(k)}(y)}{k!}(\cdot - y)^k$ is the $q-$th Taylor remainder of f at y.

The following result was proved for the considered Cantor-type sets with $N_n = N$.

Proposition 3.4 *([2]) If* $\liminf \alpha_n > N$, *then* $K_N^{(\alpha_n)}$ *does not have the extension property. If* $\limsup \alpha_n < N$, *then* $K_N^{(\alpha_n)}$ *has the extension property.*

Corollary 3.5 *For a compact set* $K_N^{(\alpha_n)}$, *let the limit* $\alpha = \lim \alpha_n$ *exist and be not equal to* N. *Then* $K_N^{(\alpha_n)}$ *has the extension property if and only if* $\lambda_0(K_N^{(\alpha_n)}) > 1$.

In general, the logarithmic dimension cannot be used for characterization of the extension property. What is more, recently it was shown in [13] that there is no such characterization in terms of Hausdorff measures, Hausdorff contents, their densities or related characteristics.

On the other hand, the logarithmic dimension is quite suitable to describe the diametral dimension of the space $\mathcal{E}(K)$, see Section 4 in [2] for more details. In particular,

Corollary 3.6 *([2]) If spaces of the type $\mathcal{E}(K_N^\alpha)$ are isomorphic, then the corresponding compact sets have the same logarithmic dimension.*

4. Polynomial bases for small Cantor-type sets

The Grothendieck problem of the existence of a basis in a nuclear Fréchet (NF) space was open for a long time. In 1974 the first example of a NF space without basis was found in [15]. After this many other examples of nuclear spaces without basis were presented, but all of them are either artificial as in [3], [17] or non-metrizable [8]. Therefore, no natural NF space of functions without basis has been found so far. This explains the interest to basis problem in concrete functional spaces.

Any Schauder basis in a NF space is absolute, therefore in order to construct a basis in such a space, it is enough to present a biorthogonal system satisfying the following Dynin-Mityagin criterion ([14]).

Let E be a nuclear Fréchet space with topology given by an increasing sequence of norms $(||\cdot||_p)_{p=1}^\infty$. Let E' be the topological dual space and $|\cdot|_{-q}$ denote the dual norm, that is, for $\xi \in E$, $|\xi|_{-q} := \sup\{|\xi(f)|, \|f\|_q \leq 1\}$. Suppose $\{e_n \in E, \xi_n \in E', n \in \mathbb{N}\}$ is a biorthogonal system such that the set of functionals $(\xi_n)_{n=1}^\infty$ is total over E. The last means that $f = 0$ if $\xi_n(f) = 0$ for all n. Assume that for every p there exist a q and a C such that for all n

$$\|e_n\|_p \cdot |\xi_n|_{-q} \leq C. \tag{4.1}$$

Then the system $(e_n, \xi_n)_{n=1}^\infty$ is an absolute basis in E.

Given a perfect compact set $K \subset \mathbb{R}$ and a sequence of distinct points $(x_k)_1^\infty \subset K$, let $e_0 = 1$ and $e_n(x) = \prod_1^n (x - x_k)$ for $n \in \mathbb{N}$. By $\xi_n(f)$ we denote the $n-$th divided difference $[x_1, x_2, \dots, x_{n+1}]f$ of a function f. By the properties of divided differences, see for instance [7], the system $(e_n, \xi_n)_{n=1}^\infty$ is biorthogonal. If, in addition, the sequence $(x_k)_1^\infty$ is dense in K, then the functionals $\xi_n, n = 0, 1, \dots,$ are total over $\mathcal{E}(K)$.

Our claim is that the space $\mathcal{E}(K_{(N_n)}^{(\alpha_n)})$ possesses an interpolating Faber basis provided $\lambda_0(K_{(N_n)}^{(\alpha_n)}) < 1$. Recall that a polynomial basis $(P_n)_{n=0}^\infty$ in a function space X is called a Faber basis if $\deg P_n = n$ for all n. The task is to find a sequence $(x_k)_1^\infty \subset K_{(N_n)}^{(\alpha_n)}$ such that the corresponding system $(e_n, \xi_n)_{n=0}^\infty$ satisfies (4.1). When the sequence will be determined, set $Z_M := (x_k)_1^M$. As in Theorem 2.3, we write K instead of $K_{(N_n)}^{(\alpha_n)}$.

Let us first consider the representation of numbers in mixed numerical bases. Let A_n denote the number of intervals in E_n, so $A_0 = 1$ and $A_n = N_1 \cdots N_n$.

Lemma 4.1 *Suppose that $A_n \leq M < A_{n+1}$. Then M has a unique representation in the form $M = \sum_{j=0}^n k_j A_j$ with $1 \leq k_n \leq N_{n+1} - 1$ and $0 \leq k_j \leq N_{j+1} - 1$ for $0 \leq j \leq n - 1$.*

Proof Indeed, let us subtract from M the value A_n several times in succession while the result is nonnegative. We can do this k_n times with $k_n \leq N_{n+1} - 1$. For the remainder we have $0 \leq M - k_n A_n < A_n$ and the same reasoning applies to k_j for $j = n - 1, n - 2, \dots, 0$. □

We compose the desired sequence $(x_k)_1^\infty$ from all left endpoints of basic intervals. Write each basic interval as $I_{j,n} = [a_{j,n}, b_{j,n}]$. Let x be a left endpoint of some basic interval. Then there exists a minimal number s (the *type* of x) such that x is the endpoint of some $I_{j,m}$ for every $m \geq s$. By X_n we denote all points of the type n. Hence, $X_0 := \{0\}, X_1$ contains $N_1 - 1$ points $a_{i,1} = (i-1)(\ell_1 + h_0)$ for $2 \leq i \leq N_1$. Continuing in this manner, we obtain

$$X_2 = \{(i-1)(\ell_1 + h_0) + (j-1)(\ell_2 + h_1) \text{ with } 1 \leq i \leq N_1, 2 \leq j \leq N_2\} \tag{4.2}$$

and, in general, $X_n = \{(i_1 - 1)(\ell_1 + h_0) + (i_2 - 1)(\ell_2 + h_1) + \cdots + (i_n - 1)(\ell_n + h_{n-1})\}$, where $1 \leq i_j \leq N_j$ for $1 \leq j \leq n-1$ and $2 \leq i_n \leq N_n$. We see that X_n contains $A_n - A_{n-1}$ points $a_{j,n}$ with $j \neq k\, N_n + 1$ for $0 \leq k \leq A_{n-1} - 1$. Set $Y_n = \cup_{k=0}^n X_k$. Then $\#(Y_n) = A_n$. Here and below, $\#(Z)$ denotes the cardinality of a finite set Z. If Z is fixed then for brevity $\nu_{j,s} := \#(I_{j,s} \cap Z)$. Also, for each $x \in \mathbb{R}$, by $d_k(x, Z), k = 1, 2, \ldots, \#(Z)$, we denote the distances $|x - z_{j_k}|$ from x to points of Z arranged in the nondecreasing order.

Let us arrange points from $\cup_{k=0}^\infty X_k$ in order, including successively points of all types in ascending order. For points of the same type, the following procedure is used to ensure a uniform distribution of points on K. First $x_1 = 0$. The points from X_1 we arrange in their natural order: $x_k = (k-1)(\ell_1 + h_0)$ for $2 \leq k \leq N_1$. Now each $I_{j,1}$ contains exactly one point from Z_{A_1}. To enumerate points from X_2, we fix the value $j = 2$ in (4.2) and consider $i = 1, 2, \ldots, N_1$. Then the same we do for $j = 3, 4, \ldots, N_2$. This gives $x_{N_1+1} = \ell_2 + h_1 = a_{2,2}$, $x_{N_1+2} = \ell_1 + h_0 + \ell_2 + h_1 = a_{N_1+2,2}$, so x_{N_1+k} is the left endpoint of the second subinterval $I_{j,2}$ of $I_{k,1}$ for $1 \leq k \leq N_1$. Next, $x_{2N_1+k} = (k-1)(\ell_1 + h_0) + 2(\ell_2 + h_1)$ is $a_{j,2}$ of the third $I_{j,2}$ subinterval of $I_{k,1}$ for $1 \leq k \leq N_1$, etc. Maximal possible values $i = N_1, j = N_2$ give the point $x_k = 1 - \ell_2$ with the index $k = N_1 + (N_2 - 1)N_1 = A_2$. We note that, if $A_1 \leq M < A_2$, then for the set Z_M the condition $\nu_{j,2} \in \{0, 1\}$ is valid for each j with $1 \leq j \leq A_2$, whereas $\nu_{i,1} \in \{1, \ldots, N_2\}$ for $1 \leq i \leq A_1$.

We use the same lexicographic order to list points from X_n for $n \geq 3$: first fix the values $i_n = 2, i_{n-1} = \cdots = i_2 = 1$, and consider $i_1 = 1, 2, \ldots, N_1$, after this enlarge i_2 by 1, take again $i_1 = 1, 2, \ldots, N_1$, etc. Maximal x_k in X_n is $1 - \ell_n$ with $k = A_n$. Clearly, $(x_n)_1^\infty$ is dense in K. We warn the reader that in [12] a different, more symmetric distribution of points x_k was used. Nevertheless, as in [12] and [13], the points Z_M are distributed uniformly on K in the following sense: for each $s \in \mathbb{N}$ and $i, j \in \{1, 2, \ldots, A_s\}$ we have

$$|\nu_{j,s} - \nu_{i,s}| \leq 1, \tag{4.3}$$

so any two intervals of the same level contain the same number of points from Z_M or, perhaps, one of the intervals contains one extra point x_k, compared to another interval.

Suppose $A_n \leq M < A_{n+1}$. Then $M = k_n\, A_n + r_n$ with $1 \leq k_n \leq N_{n+1} - 1$ and $0 \leq r_n < A_n$. There are A_n intervals of $n-$th level. Hence, for each j we have $k_n \leq \nu_{j,n} \leq k_n + 1$. Lemma 4.1 yields the representation $M = (k_n\, N_n + k_{n-1})A_{n-1} + r_{n-1}$ with $0 \leq r_{n-1} < A_{n-1}$. Therefore, $k_n\, N_n + k_{n-1} \leq \nu_{j,n-1} \leq k_n\, N_n + k_{n-1} + 1$. Similarly, for $0 \leq s \leq n-1$ and $1 \leq j \leq A_s$ we have

$$k_n\, N_n \cdots N_{s+1} + k_{n-1}\, N_{n-1} \cdots N_{s+1} + \cdots + k_s \leq \nu_{j,s} \leq k_n\, N_n \cdots N_{s+1} + \cdots + k_s + 1. \tag{4.4}$$

In the case of bounded sequence, let $N_k \leq N$ for all k, we have $1 \leq \nu_{j,n} \leq N$ and, for $s < n$,

$$N_n \cdots N_{s+1} \leq \nu_{j,s} \leq N^{n-s+1}.$$

Our next objective is to associate with a given M a set $(m_k)_{k=0}^n$ of natural numbers which will be used in estimations of $||e_M||_p$ and $|\xi_M|_{-q}$. For each $x \in K$ we have the chain of basic intervals containing x: $x \in I_{j,n} \subset I_{j_1,n-1} \subset \cdots \subset I_{j_n,0} = [0,1]$.

Let $m_n(x) = \nu_{j,n} = \#(Z_M \cap I_{j,n})$ and $m_k(x) = \nu_{j_{n-k},k} - \nu_{j_{n-k-1},k+1}$ for $0 \le k \le n-1$, so $m_k(x)$ is the number of zeros of e_M in $I_{j_{n-k},k}$ which do not belong to $I_{j_{n-k-1},k+1}$. Then $|e_M(x)| = \prod_{i=1}^M d_i(x, Z_M)$ with $d_i(x, Z_M) \le \ell_n$ for $1 \le i \le m_n(x)$, $d_i(x, Z_M) \le \ell_{n-1}$ for the next $m_{n-1}(x)$ values of i, etc. This gives

$$|e_M(x)| \le \ell_n^{m_n(x)} \cdots \ell_0^{m_0(x)}. \tag{4.5}$$

Let us find minimal possible values of $(m_k)_{k=0}^n$ for which (4.5) is valid for all $x \in K$. Since $k_n A_n \le M < (k_n+1)A_n$, at least one $I_{j,n}$ contains exactly k_n points from Z_M. Hence we must take $m_n = k_n$. Since $(k_n N_n + k_{n-1}) A_{n-1} \le M < (k_n N_n + k_{n-1} + 1) A_{n-1}$, there is $I_{j,n-1}$ containing exactly $k_n N_n + k_{n-1}$ points from Z_M. For at least of one of its subintervals $I_{j,n}$ we have $\#(Z_M \cap I_{j,n}) = k_n$. It follows that $m_{n-1} = k_n(N_n - 1) + k_{n-1}$. Continuing in this manner, we obtain for $0 \le s \le n-1$ the representation

$$m_s = k_n N_n \cdots N_{s+2}(N_{s+1} - 1) + \cdots + k_{s+1}(N_{s+1} - 1) + k_s. \tag{4.6}$$

Then for each $x \in K$ we have

$$|e_M(x)| = \prod_{i=1}^M d_i(x, Z_M) \le \ell_n^{m_n} \cdots \ell_0^{m_0}, \tag{4.7}$$

where the set $(m_k)_{k=0}^n$ does not depend on x. It is easy to check that $m_n + \cdots + m_0 = M$, so $\ell_n^{m_n} \cdots \ell_0^{m_0}$ is a product of M nondecreasing terms:

$$\ell_n^{m_n} \cdots \ell_0^{m_0} = \prod_{k=1}^M \rho_k \quad \text{where} \quad \rho_1 \le \rho_2 \le \cdots \le \rho_M. \tag{4.8}$$

Lemma 4.2 *Suppose $N_n \le N$ for all n. Let M be as in Lemma 4.1, $m_n = k_n$ and $(m_s)_{s=0}^{n-1}$ be given by (4.6). Then for any natural numbers r, s with $2 \le r \le r+s \le n$ we have*

$$\sum_{j=r}^{r+s} m_{n-j} \le N^{s+3} m_{n-r+1}.$$

Proof By Lemma 4.1, $k_n \ge 1$ and $k_j \ge 0$ for $0 \le j \le n-1$. This gives

$$N_n \cdot N_{n-1} \cdots N_{n-r+2}(N_{n-r+1} - 1) \le m_{n-r}. \tag{4.9}$$

Substituting the maximal possible values $k_j = N_{j+1} - 1$ into (4.6) yields

$$m_{n-r} \le N_{n+1} \cdot N_n \cdots N_{n-r+2}(N_{n-r+1} - 1). \tag{4.10}$$

We note that (4.10) is valid for $r = 1$ as well. By (4.10),

$$\sum_{j=r}^{r+s} m_{n-j} \le N_{n+1} \cdot N_n \cdots N_{n-r+3}[N_{n-r+2}(N_{n-r+1} - 1) + \cdots + N_{n-r+2} \cdots N_{n-r-s+2}(N_{n-r-s+1} - 1)].$$

Here, the sum in square brackets does not exceed N^{s+2}, as is easy to check. Hence,

$$\sum_{j=r}^{r+s} m_{n-j} \le N^{s+3} N_n \cdots N_{n-r+3}.$$

On the other hand, by (4.9), $m_{n-r+1} \ge N_n \cdots N_{n-r+3}$ as $N_{n-r+2} \ge 2$. □

Lemma 4.3 *Let $A_n \le M < A_{n+1}$ and $p < M$. Then $||e_M||_p \le C_p M^p \prod_{k=p+1}^M \rho_k$, where C_p does not depend on M.*

Proof The $i-$th derivative of e_M at x is a sum of $M!/(M-i)!$ products, where each product contains $M-i$ terms of the type $x-x_j$. Hence, $|e_M^{(i)}(x)| \le M^i \prod_{j=i+1}^M d_j(x, Z_M) \le M^i \prod_{k=i+1}^M \rho_k$, by (4.7) and (4.8). Taking supremum over all $i \le p$ and $x \in K$ we get $|e_M|_p \le M^p \prod_{k=p+1}^M \rho_k$.

As for the norms $||e_k||_p$, by (2.1), we can repeat the reasoning from the proof of Theorem 1 in [12], see page 354. □

We proceed to estimate the dual norms. For each $x_r \in Z_M$ we have $x_r \in I_{i,n} \subset I_{i_1,n-1} \subset \cdots \subset I_{i_n,0} = [0,1]$ and

$$|e'_M(x_r)| = \prod_{j \ne r} |x_r - x_j| = \prod_{j=2}^{M} d_j(x_r, Z_M) \ge h_n^{m'_n(x_r)} \cdots h_0^{m'_0(x_r)}. \tag{4.11}$$

where $m'_k(x_r)$ is the number of zeros of e_M (except the point x_r) in $I_{i_{n-k},k}$ which do not belong to $I_{i_{n-k-1},k+1}$. Thus, $(m'_k(x_r))_{k=0}^n$ are natural numbers except perhaps $m'_n(x_r)$ which is 0 if $I_{i,n} \cap Z_M = \{x_r\}$.

We search for maximal possible values of $(m'_k)_{k=0}^n$ for which (4.11) is valid for all $x_r \in Z_M$. Since $k_n \le \nu_{j,n} \le k_n + 1$ for all j and we remove x_r from consideration, $m'_n = \max \nu_{j,n} - 1 \le k_n = m_n$. In the next step, $m'_{n-1} = (\nu_{i_1,n-1} - 1) - m'_n$ with $\nu_{i_1,n-1} \le k_n N_n + k_{n-1} + 1$. Hence, $m'_{n-1} \le (k_n - 1) N_n + k_{n-1} = m_{n-1}$. Reapplying this argument yields $m'_k \le m_k$ for $0 \le k \le n$ and the following uniform with respect to x_r bound

$$|e'_M(x_r)| \ge h_n^{m_n} \cdots h_0^{m_0}. \tag{4.12}$$

Given any product $\prod_{j=1}^N \lambda_j$ with $\lambda_j \ge 0$ and $q < N$, by $(\prod_{j=1}^N \lambda_j)_q$ we denote this product without q smallest terms.

Lemma 4.4 *Suppose $A_n \le M < A_{n+1}, 1 \le q < M$. Then $|\xi_M|_{-q} \le C_q 2^M ((h_n \cdot h_n^{m_n} \cdots h_0^{m_0})_q)^{-1}$, where C_q does not depend on M.*

Proof To estimate the dual q-th norm of ξ_M we enumerate the points $(x_k)_1^{M+1}$ in increasing order and denote the rearranged set by $(y_k)_1^{M+1}$. Then $\xi_M(f) = [y_1, \ldots, y_{M+1}]f$. By (1) in [11], see also (2) in [12],

$$|\xi_M|_{-q} \le C_q 2^M \left(\min \prod_{k=q+1}^{M} |y_{a(k)} - y_{b(k)}| \right)^{-1}, \tag{4.13}$$

where minimum is taken over all j with $1 \le j \le M+1-q$ and all possible chains of strict embeddings $[y_j, \ldots, y_{j+q}] \subset \cdots \subset [y_1, \ldots, y_{M+1}]$. Here, $[y_j, \ldots, y_{j+q}] = [y_{a(q+1)}, \ldots, y_{b(q+1)}] \subset [y_{a(q+2)}, \ldots, y_{b(q+2)}] \subset \ldots \subset$

$[y_{a(M)}, \dots, y_{b(M)}] = [y_1, \dots, y_{M+1}]$ with $a(k+1) = a(k)$, $b(k+1) = b(k)+1$, or $a(k+1) = a(k)-1$, $b(k+1) = b(k)$. Let the minimal product Π in (4.13) be realized by $[y_{j_0}, \dots, y_{j_0+q}]$. We note that at least one point from the pair y_{j_0}, y_{j_0+q} belongs to Z_M. Without loss of generality let $y_{j_0} \in Z_M$. In each embedding of $[y_{j_0}, \dots, y_{j_0+q}]$ into larger interval $[y_a, \dots, y_b]$ some new endpoint, let for instance y_a, appears. Since $y_b - y_a \geq |y_{j_0} - y_a|$, we obtain $\Pi = \prod_{k=q+1}^{M} |y_{a(k)} - y_{b(k)}| \geq (\prod_{k=1,k\neq j_0}^{M+1} |y_{j_0} - y_k|)_q$. The last product represent largest $M-q$ terms of $|e'_{M+1}(y_{j_0})|$. Here, $|e'_{M+1}(y_{j_0})| = |e'_M(y_{j_0})| \cdot |y_{j_0} - x_{M+1}|$ with $|y_{j_0} - x_{M+1}| \geq \ell_{n+1} + h_n > h_n$, since $M+1 \leq A_{n+1}$. Applying (4.12) yields the desired result. □

From now on, we assume that the sequence $(N_n)_{n=1}^{\infty}$ is bounded. We present a Faber basis in the space $\mathcal{E}(K^{(\alpha_n)}_{(N_n)})$ for two cases:

1) $\alpha_n \geq N_n$ for all n. The corresponding result is a direct generalization of Theorem 1 from [12]. Here, $\lambda_0(K^{(\alpha_n)}_{(N_n)}) \leq 1$ but perhaps $\lim_n \lambda_n$ does not exist.

2) There exists $\lim_n \lambda_n$ which is smaller than 1.

Theorem 4.5 *Let $N_n \leq N$ for all n. Suppose that for a set $K^{(\alpha_n)}_{(N_n)}$ either $\alpha_n \geq N_n$ for all n or there exists $\lim_n \lambda_n < 1$. Then the sequence $(e_M)_{M=0}^{\infty}$ is a Schauder basis in the space $\mathcal{E}(K^{(\alpha_n)}_{(N_n)})$.*

Proof Given p, we need to find q and C such that for all M

$$\|e_M\|_p \cdot |\xi_M|_{-q} \leq C. \tag{4.14}$$

Let us fix any $p \in \mathbb{N}$ and take $q = p(1 + N^{w+3})$, where $w = w(N)$ will be specified later. We can consider only large enough M since otherwise (4.14) is valid with an appropriate choice of C. Hence, we can assume that M is so large that we can use above lemmas. Fix M. Let $A_n \leq M < A_{n+1}$.

Let us first apply Lemma 4.4 to the case of bounded sequence $(N_n)_{n=0}^{\infty}$. By (2.1), we have $\ell_k \leq (2N_{k+1} - 1)h_k$. It follows that $h_k > (2N)^{-1}\ell_k$ for all k and $|\xi_M|_{-q} \leq C_q (4N)^M (\prod_{k=q}^{M} \rho_k)^{-1}$, by (4.8). Thus there is a constant C_0 such that

$$\|e_M\|_p \cdot |\xi_M|_{-q} \leq C_0 \cdot M^p \, (4N)^M \prod_{k=p+1}^{q-1} \rho_k. \tag{4.15}$$

Given p, take u such that $m_n + \dots + m_{n-u+2} < p \leq m_n + \dots + m_{n-u+1}$. Consider the product from (4.8) in more detail:

$$\prod_{k=1}^{M} \rho_k = \underbrace{\ell_n \cdots \ell_n}_{m_n} \cdots \underbrace{\ell_{n-u+1} \cdots \ell_{n-u+1}}_{m_{n-u+1}} \underbrace{\ell_{n-u} \cdots \ell_{n-u}}_{m_{n-u}} \cdots \underbrace{\ell_{n-u-w+1} \cdots \ell_{n-u-w+1}}_{m_{n-u-w+1}} \cdots \underbrace{\ell_0 \cdots \ell_0}_{m_0}.$$

Here, $m_n + m_{n-1} + \dots + m_{n-u-w+1} < p + \sum_{j=u-1}^{u+w-1} m_{n-j} \leq p + N^{w+3} m_{n-u+2}$, by Lemma 4.2. But $m_{n-u+2} < p$. Hence, the sum above does not exceed $q-1$ and interval $[\rho_{p+1}, \dots \rho_{q-1}]$ covers

$$\underbrace{\ell_{n-u} \cdots \ell_{n-u}}_{m_{n-u}} \cdots \underbrace{\ell_{n-u-w+1} \cdots \ell_{n-u-w+1}}_{m_{n-u-w+1}}.$$

Therefore, $\prod_{k=p+1}^{q-1}\rho_k \le \ell_{n-u}^{m_{n-u}} \cdots \ell_{n-u-w+1}^{m_{n-u-w+1}}$. The last product is ℓ_1^κ with $\kappa = m_{n-u}\alpha_1\cdots\alpha_{n-u} + \cdots + m_{n-u-w+1}\alpha_1\cdots\alpha_{n-u-w+1}$. It remains to find a constant C such that for all M

$$M^p\,(4N)^M \ell_1^\kappa \le C.$$

Recall that $M < A_{n+1} \le N\,A_n$; therefore, the desired inequality reduces to

$$p\,\log(N\,A_n) + A_n\,N\log(4N) \le C + \kappa\cdot\log(1/\ell_1). \tag{4.16}$$

By (4.9), $m_{n-r} \ge N_n\cdot N_{n-1}\cdots N_{n-r+2} \ge (N)^{-1}N_n\cdot N_{n-1}\cdots N_{n-r+1}$. For this reason,

$$m_{n-r}\alpha_1\cdots\alpha_{n-r} \ge N^{-1}N_1\cdots N_n\cdot\frac{\alpha_1\cdots\alpha_{n-r}}{N_1\cdots N_{n-r}}.$$

Hence,

$$\kappa \ge N^{-1}A_n\sum_{j=u}^{u+w-1}\frac{\alpha_1\cdots\alpha_{n-j}}{N_1\cdots N_{n-j}}.$$

In the first case, when $\alpha_n \ge N_n$ for all n, we have $\kappa \ge N^{-1}A_n\,w$. We see that the choice $w = N^3$ provides (4.16).

In the second case, when $\lim_n\lambda_n = \lambda_0 < 1$, let us take $\tilde{n}$ such that $\lambda_n \le 1$ for $n \ge \tilde{n}$. By(2.2), $\frac{\alpha_1\cdots\alpha_{n-j}}{N_1\cdots N_{n-j}} = (N_1\cdots N_{n-j})^{\frac{1-\lambda_{n-j}}{\lambda_{n-j}}} \ge 1$ for large enough n and bounded j. Here, as above, $\kappa \ge N^{-1}A_n\,w$ and we can take the same w. This gives (4.16) and (4.14). □

Remarks. 1. The same reasoning applies to the case when $\lambda_n \searrow 1$ so fast that the sequence $(\lambda_n - 1)\log A_n$ is bounded.
2. We think that for the general case, the method of local interpolations, see [12] and [13], can be used to construct topological (in general, not Faber) bases in $\mathcal{E}(K_{(N_n)}^{(\alpha_n)})$, see question on page 237 in [2].

Acknowledgments

The authors thank the referee for valuable comments and suggestions on improving the presentation of the paper. The research was partially supported by TÜBİTAK (Scientific and Technological Research Council of Turkey), Project 119F023.

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